CHARACTERIZING COMPACT CLIFFORD SEMIGROUPS THAT EMBED INTO CONVOLUTION AND FUNCTOR-SEMIGROUPS

T. BANAKH, M. CENCELJ, O. HRYNIV, AND D. REPOVŠ

Abstract. We study algebraic and topological properties of the convolution semigroups of probability measures on topological groups and show that a compact Clifford topological semigroup S embeds into the convolution semigroup $P(G)$ over some topological group G if and only if S embeds into the semigroup $\exp(G)$ of compact subsets of G if and only if S is an inverse semigroup and has zero-dimensional maximal semilattice. We also show that such a Clifford semigroup S embeds into the functor-semigroup $F(G)$ over a suitable compact topological group G for each weakly normal monadic functor F in the category of compacta such that $F(G)$ contains a G-invariant element (which is an analogue of the Haar measure on G).

1. Introduction

According to [6] (and [17]) each (commutative) semigroup S embeds into the global semigroup $\Gamma(G)$ over a suitable (abelian) group G. The global semigroup $\Gamma(G)$ over G is the set of all non-empty subsets of G endowed with the semigroup operation $(A, B) \mapsto AB = \{ab : a \in A,\ b \in B\}$. If G is a topological group, then the global semigroup $\Gamma(G)$ contains a subsemigroup $\exp(G)$ consisting of all non-empty compact subsets of G and carrying a natural topology which makes it a topological semigroup. This is the Vietoris topology generated by the sub-base consisting of the sets

$$U^+ = \{K \in \exp(S) : K \subset U\} \text{ and } U^- = \{K \in \exp(S) : K \cap U \neq \emptyset\}$$

where U runs over open subsets of S. Endowed with the Vietoris topology the semigroup $\exp(G)$ will be referred to as the *hypersemigroup* over G (because its underlying topological space is the hyperspace $\exp(G)$ of G, see [14]). The problem of detecting topological semigroups embeddable into the hypersemigroups over topological groups has been considered in the literature, see [6].

This problem was resolved in [5] for the class of Clifford compact topological semigroups: such a semigroup S embeds into the hypersemigroup over a topological group if and only if the set E of idempotents of S is a zero-dimensional commutative subsemigroup of S. This characterization implies the result of [7] that the closed interval $[0, 1]$ with the operation of the minimum does not embed into the hypersemigroup over a topological group.

Date: November 6, 2008.

2000 *Mathematics Subject Classification.* 20M18, 20M30, 22A15, 43A05, 54B20, 54B30.

Key words and phrases. Convolution semigroup, global semigroup, hypersemigroup, Clifford semigroup, regular semigroup, topological group, Radon measure, weakly normal monadic functor.

This research was supported by the Slovenian Research Agency grants P1-0292-0101-04, J1-9643-0101 and BI-UA/07-08-001.

We recall that a semigroup S is Clifford if S is the union of its subgroups. We say that a topological semigroup S_1 embeds into another topological semigroup S_2 if there is a semigroup homomorphism $h : S_1 \to S_2$ which is a topological embedding.

In this paper we shall apply the already mentioned result of [5] and shall characterize Clifford compact semigroups embeddable into the convolution semigroups $P(G)$ over topological groups G. The convolution semigroup $P(G)$ consists of probability Radon measures on G and carries the $*$-weak topology generated by the sub-base $\{\mu \in P(G) : \mu(U) > a\}$ where $a \in \mathbb{R}$ and U runs over open subsets of G. A measure μ defined on the σ-algebra of Borel subsets of G is called *Radon* if for every $\varepsilon > 0$ there is a compact subset $K \subset G$ with $\mu(K) > 1 - \varepsilon$. The semigroup operation on $P(G)$ is given by the convolution measures. We recall that the *convolution* $\mu * \nu$ of two measures μ, ν is the measure assigning to each bounded continuous function $f : G \to \mathbb{R}$ the value of the integral $\int_{\mu*\nu} f = \int_\mu \int_\eta f(xy)dydx$. For more detail information on the convolution semigroups, see [10], [11].

The following theorem is the principal result of this paper.

Theorem 1.1. *For any Clifford compact topological semigroup S the following assertions are equivalent:*

1. *S embeds into the hypersemigroup $\exp(G)$ over a topological group G;*
2. *S embeds into the convolution semigroup $P(X)$ over a topological group G;*
3. *The set E of idempotents of S is a zero-dimensional commutative subsemigroup of S.*

This theorem will be applied to a characterization of Clifford compact topological semigroups embeddable into the hyperpsemigroups or convolution semigroups over topological groups G belonging to certain varieties of topological groups. A class $\mathcal{G}$ of topological groups is called a *variety* if it is closed under arbitrary Tychonov products, and taking closed subgroups, and quotient groups by closed normal subgroups.

Theorem 1.2. *Let $\mathcal{G}$ be a non-trivial variety of topological groups. For a Clifford compact topological semigroup S the following assertions are equivalent:*

1. *S embeds into the hypersemigroup $\exp(G)$ over a topological group $G \in \mathcal{G}$;*
2. *S embeds into the convolution semigroup $P(G)$ over a topological group $G \in \mathcal{G}$;*
3. *The set E of idempotents is a zero-dimensional commutative subsemigroup of S and all closed subgroups of S belong to the class $\mathcal{G}$.*

In fact, the equivalence of the first and last statement in Theorems 1.1 and 1.2 was proved in [5, Th 3,4] so it remains to prove the equivalence of the assertions (1) and (2). This will be done in Proposition 1.3, which says that for each topological group G the semigroups $\exp(G)$ and $P(G)$ have the same regular subsemigroups. We recall that a semigroup S is called *regular* if each element $x \in S$ is *regular* in the sense that $xyx = x$ for some $y \in S$. An element $x \in S$ is called (*uniquely*) *invertible* if there is a (unique) element $x^{-1} \in S$ (called the *inverse* of x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. A semigroup S is called *inverse* if each element of S is uniquely invertible. By [8, 1.17], [12, II.1.2] a semigroup S is inverse if and only if it is regular and the set E of idempotents of S is a commutative subsemigroup of S. An inverse semigroup S is Clifford if and only if $xx^{-1} = x^{-1}x$ for all $x \in S$. In this case S decomposes into the union $S = \bigcup_{e\in E} H_e$ of the maximal subgroups $H_e = \{x \in S : xx^{-1} = e = x^{-1}x\}$ of S parametrized by idempotents e of S.

The following proposition shows that the semigroups $\exp(G)$ and $P(G)$ over a topological group G have the same regular subsemigroups (which are necessarily topological inverse semigroups). Moreover, regular subsemigroups of $\exp(G)$ or $P(G)$ have many specific topological and algebraic features.

We recall that a topological semigroup S is called a *topological inverse semigroup* if S is an inverse semigroup and the inversion map $(\cdot)^{-1}: S \to S$, $(\cdot)^{-1}: x \mapsto x^{-1}$ is continuous. The set E of idempotents of a topological inverse semigroup S is a closed commutative subgroup of S called the *idempotent semilattice* of S. We say that two idempotents $e, f \in E$ are *incomparable* if their product ef differs from e and f. Two elements x, y of an inverse semigroup S are called *conjugate* if $x = zyz^{-1}$ and $y = z^{-1}xz$ for some element $z \in S$. For any idempotent $e \in E$ let ${\uparrow}e = \{f \in E : ef = e\}$ denote the principal filter of E. A topological space X is called *totally disconnected* if for any distinct points $x, y \in X$ there is a closed-and-open subset $U \subset X$ containing x but not y.

Proposition 1.3. *Let G be a topological group. A topological regular semigroup S embeds into $P(G)$ if and only if S embeds into* $\exp(G)$. *If the latter happens, then*

1. *S is a topological inverse semigroup;*
2. *The idempotent semilattice E of S has totally disconnected principal filters ${\uparrow}e$, $e \in E$;*
3. *An element $x \in S$ is an idempotent if and only if x^2x^{-1} is an idempotent;*
4. *Any distinct conjugated idempotents of S are incomparable.*

This proposition allows one to construct many examples of topological regular semigroups non-embeddable into the hypersemigroups or convolution semigroups over a topological groups. The first two assertions of this proposition imply the result of [7] to the effect that non-trivial rectangular semigroups and connected topological semilattices do not embed into the hypersemigroup $\exp(G)$ over a topological group G. The last two assertions imply that the semigroups $\exp(G)$ and $P(G)$ do not contain Brandt semigroups and bicyclic semigroups. By a *Brandt semigroup* we understand a semigroup of the form $B(H, I) = I \times H \times I \cup \{0\}$ where H is a group, I is a non-empty set, and the product $(\alpha, h, \beta) * (\alpha, h', \beta')$ of two non-zero elements of $B(H, I)$ is equal to (α, hh', β') if $\beta = \alpha'$ and 0 otherwise. A *bicyclic semigroup* is a semigroup generated by two elements p, q with the relation $qp = 1$. Brandt semigroups and byciclic semigroups play an important role in the structure theory of inverse semigroups, see [12].

In fact, the semigroups $\exp(G)$ and $P(G)$ are special cases of the so-called functor-semigroups introduced by Teleiko and Zarichnyi [14]. They observed that any weakly normal monadic functor $F : \mathcal{C}omp \to \mathcal{C}omp$ in the category of compact Hausdorff spaces lifts to the category of compact topological semigroups, which means that for any compact topological semigroup X the space FX possesses a natural semigroup structure. The semigroup operation $*$ on FX can be defined by the following formula

$$a * b = Fp(a \otimes b) \text{ for } a, b \in FX$$

where $p : X \times X \to X$ is the semigroup operation of X and $a \otimes b \in F(X \times X)$ is the tensor product of the elements $a, b \in FX$, see [14, §3.4].

Therefore we actually consider in this paper the following general problem:

Problem 1.4. *Given a weakly normal monadic functor $F : \mathcal{C}omp \to \mathcal{C}omp$, find a characterization of compact (regular, inverse, Clifford) topological semigroups embeddable into the semigroup FX over a compact topological group X. Given a compact topological group X describe invertible elements and idempotents of the semigroup FX.*

Observe that for the functors exp and P the answer to the first part of this problem is given in Theorem 1.1. Functor-semigroups induced by the functors G of inclusion hyperspaces and λ of superextension have been studied in [9], [4], and [1]–[3].

In fact, Theorem 1.2 also can be partly generalized to some monadic functors F (including the functors exp, P, G and λ). Given a compact topological group G let us define an element $a \in F(G)$ to be *G-invariant* if $g * a = a = a * g$ for every $g \in G$. Here we identify G with a subspace of $F(G)$ (which is possible because F, being weakly normal, preserves singletons). A G-invariant element in $F(G)$ exists for the functors exp, P, λ, and G. For the functors exp and P a G-invariant element on $F(G)$ is unique: it is $G \in \exp(G)$ and the Haar measure on G, respectively.

Theorem 1.5. *Let $F : \mathcal{C}omp \to \mathcal{C}omp$ be a weakly normal monadic functor such that for every compact topological group G the semigroup $F(G)$ contains a G-invariant element. Each Clifford compact topological inverse semigroup S with zero-dimensional idempotent semilattice E embeds into the functor-semigroup $F(G)$ over the compact topological group $G = \prod_{e\in E} \widetilde{H}_e$ where each $\widetilde{H}_e$ is a non-trivial compact topological group containing the maximal subgroup $H_e \subset S$ correspnding to an idempotent $e \in E$ of S.*

Proof. By Theorem 3 of [5], each Clifford compact topological inverse semigroup S with zero-dimensional idempotent semilattice E embeds into the product $\prod_{e\in E} H_e^0$, where H_e^0 stands for the extension of the maximal subgroup H_e by an isolated point $0 \notin H_e$ such that $x0 = 0x = 0$ for all $x \in H_e$. For every idempotent $e \in E$, fix a non-trivial compact topological group $\widetilde{H}_e$ containing H_e. By our hypothesis, the space $F(\widetilde{H}_e)$ contains an $\widetilde{H}_e$-invariant element $z_e \in F(\widetilde{H}_e)$. Then H_e^0 can be identified with the closed subsemigroup $H_e \cup \{z_e\}$ of $F(\widetilde{H}_e)$ and the product $\prod_{e\in E} H_e^0$ can be identified with a subsemigroup of the product $\prod_{e\in E} F(\widetilde{H}_e)$. By [14, p.126], the latter product can be identified with a subspace (actually a subsemigroup) of $F(\prod_{e\in E} \widetilde{H}_e) = F(G)$, where $G = \prod_{e\in E} \widetilde{H}_e$. In this way, we obtain an embedding of S into $F(G)$. $\square$

As we have said, the functors λ of superextension and G of inclusion hyperspaces satisfy the hypothesis of Theorem 1.5. However, Proposition 1.3 is specific for the functor P and cannot be generalized to the functors λ or G.

Indeed, for the 4-element cyclic group C_4 the semigroup $\lambda(C_4)$ is isomorphic to the commutative inverse semigroup $C_4 \oplus C_2^1$, where $C_2^1 = C_2 \cup \{1\}$ is the result of attaching an external unit to the 2-element cyclic group C_2, (see [4]). On the other hand, the 12-element semigroup $C_4 \oplus C_2^1$ cannot be embedded into $\exp(C_4)$ because the set of regular elements of $\exp(C_4)$ consists of 7 elements (which are shifted subgroups of C_4). Also the commutative inverse semigroup $\lambda(C_4) \cong C_4 \oplus C_2^1$ can be embedded into $G(C_4)$ (because λ is a submonad of G) but cannot embed into $\exp(C_4)$.

2. Idempotents and invertible elements of the convolution semigroups

In this section we prove Proposition 1.3. For each topological group G the semigroups $P(G)$ and $\exp(G)$ are related via the map of the support. We recall that the *support* of a Radon measure $\mu \in P(G)$ is the closed subset

$$S_\mu = \{x \in G : \mu(Ox) > 0 \text{ for each neighborhood } Ox \text{ of } x\}$$

of G. Let 2^G denote the semigroup of all non-empty closed subsets of G endowed with the semigroup operation $A * B = \overline{AB}$. By

$$\mathrm{supp} : P(G) \to 2^G,\ \mathrm{supp} : \mu \mapsto S_\mu$$

we denote the support map.

The following proposition is well-known, see (the proof of) Theorem 1.2.1 in [10].

Proposition 2.1. *Let G be a topological group. For any measures $\mu, \nu \in P(G)$ the following holds: $S_{\mu*\nu} = \overline{S_\mu \cdot S_\nu}$. This means that the support map* $\mathrm{supp} : P(G) \to 2^G$ *is a semigroup homomorphism.*

We shall show that for any regular element μ of the convolution semigroup $P(G)$ the support S_μ is compact and thus belongs to the subsemigroup $\exp(G)$ of 2^G. First, we characterize idempotent measures on a topological group G.

A measure $\mu \in P(G)$ is called an *idempotent measure* if $\mu*\mu = \mu$. In 1954 Wendel [18] proved that each idempotent measure on a compact topological group coincides with the Haar measure of some compact subgroup. Later, Wendel's result was generalized to locally compact groups by Pym [13] and to all topological groups by Tortrat [16]. By the *Haar measure* on a compact topological group G we understand the unique G-invariant probability measure on G. It is a classical result that such a measure exists and is unique. Thus we have the following characterization of idempotent measures on topological groups:

Proposition 2.2. *A probability Radon measure $\mu \in P(G)$ on a topological group G is an idempotent of the semigroup $P(G)$ if and only if μ is the Haar measure of some compact subgroup of G.*

We shall use this proposition to describe regular elements of the convolution semigroups. To this end we apply Proposition 4 of [5] that describes regular elements of the hypersemigroups over topological groups:

Proposition 2.3 (Banakh-Hryniv)**.** *For a compact subset $K \in \exp(G)$ of a topological group G the following assertions are equivalent:*

1. *K is a regular element of the semigroup* $\exp(G)$*;*
2. *K is uniquely invertible in* $\exp(G)$*;*
3. *$K = Hx$ for some compact subgroup H of G.*

A similar description of regular elements holds for the convolution semigroup:

Proposition 2.4. *For a measure $\mu \in P(G)$ on a topological group G the following assertions are equivalent:*

1. *μ is a regular element of the semigroup $P(G)$;*
2. *μ uniquely invertible in $P(G)$;*
3. *$\mu = \lambda * x$ for some idempotent measure $\lambda \in P(G)$ and some element $x \in G$.*

Proof. Assume that μ is a regular element of $P(G)$ and $\nu \in P(G)$ is a measure such that $\mu * \nu * \mu = \mu$. The measure $\mu * \nu$, being an idempotent of $P(G)$ coincides with the Haar measure λ on some compact subgroup H of G. It follows that $\overline{S_\mu \cdot S_\nu} = S_{\mu*\nu} = S_\lambda = H$ and hence S_μ and S_ν are compact subsets of the group G. Since supp : $P(G) \to 2^G$ is a semigroup homomorphism, we get $S_\mu * S_\nu * S_\mu = S_\mu$, which means that S_μ is a regular element of the semigroup $\exp(G)$ and hence $S_\mu = \tilde{H}x$ for some compact subgroup $\tilde{H}$ and some element $x \in G$ according to Proposition 2.3.

We claim that $\tilde{H} = H$. Indeed, $H\tilde{H}x = S_\lambda S_\mu = S_{\mu*\nu}S_\mu = S_{\mu*\nu*\mu} = S_\mu = \tilde{H}x$ implies that $H \subset \tilde{H}$. Next, for any point $y \in S_\nu$ we get

$$\tilde{H}xy \subset \tilde{H}xS_\nu = S_\mu S_\nu = S_\lambda = H \subset \tilde{H}$$

which yields $xy \in \tilde{H}$ and finally $H = \tilde{H}$.

Next, we show that $\mu = \lambda * x$, which is equivalent to $\lambda = \mu * x^{-1}$. Observe that $S_{\mu*x^{-1}} = S_\mu x^{-1} = Hxx^{-1} = H$. Now the equality $\mu * x^{-1} = \lambda$ will follow as soon as we check that the measure $\mu * x^{-1}$ is H-invariant. Take any point $y \in H$ and note that

$$y * \mu * x^{-1} = y * \mu * \nu * \mu * x^{-1} = y * \lambda * \mu * x^{-1} = \lambda * \mu * x^{-1} = \mu * x^{-1},$$

which means that the measure $\mu * x^{-1}$ on H is left-invariant. Since H possesses a unique left-invariant probability measure λ, we conclude that $\mu = \lambda * x$.

Finally, we show that μ is uniquely invertible in $P(G)$. It suffices to check that the measure ν is equal to $x^{-1} * \lambda$ provided $\nu = \nu * \mu * \nu$. For this just observe that S_ν being a unique inverse of S_μ is equal to $x^{-1}H$. Then $S_{x*\nu} = xS_\nu = xx^{-1}H$. Finally, noticing that for every $y \in H$ we get

$$x * \nu * y = x * \nu * \mu * \nu * y = x * \nu * \lambda * y = x * \nu * \lambda = x * \nu,$$

which means that $x * \nu$ is a right invariant measure on H. Since λ is the unique right-invariant measure on H we also get $x * \nu = \lambda$ and hence $\nu = x^{-1} * \lambda$. □

Given a semigroup S we denote the set of regular elements of S by $\mathrm{Reg}(S)$.

Proposition 2.5. *For any topological group G, the support map*

$$\mathrm{supp} : \mathrm{Reg}(P(G)) \to \mathrm{Reg}(\exp(G))$$

is a homeomorphism.

Proof. The preceding proposition implies that the map

$$\mathrm{supp} : \mathrm{Reg}(P(G)) \to \mathrm{Reg}(\exp(G))$$

is bijective. In order to check the continuity of this map, we must prove that for any open set $U \subset G$ the preimages

$$\mathrm{supp}^{-1}(U^+) = \{\mu \in \mathrm{Reg}(P(G)) : \mathrm{supp}(\mu) \subset U\} \text{ and}$$
$$\mathrm{supp}^{-1}(U^-) = \{\mu \in \mathrm{Reg}(P(G)) : \mathrm{supp}(\mu) \cap U \neq \emptyset\}$$

are open in $P(G)$. The openness of $\mathrm{supp}^{-1}(U^-)$ follows from the observation that $\mathrm{supp}(\mu) \cap U \neq \emptyset$ if and only if $\mu(U) > 0$. To see that $\mathrm{supp}^{-1}(U^+)$ is open, fix any measure $\mu \in \mathrm{Reg}(P(G))$ with $\mathrm{supp}(\mu) \subset U$. By Proposition 2.4, $\mathrm{supp}(\mu) = Hx$ for some compact subgroup H of G and some $x \in G$. The

compactness of H allows us to find an open neighborhood V of the neutral element of G such that $HV^2HV^{-2}HV \subset Ux^{-1}$. Now consider the open neighborhood $W = \{\nu \in \mathrm{Reg}(P(G)) : \nu(HVx) > \frac{1}{2}\}$ of the measure μ. We claim that $W \subset \mathrm{supp}^{-1}(U^+)$. Indeed, given any measure $\nu \in W$ we can apply Proposition 2.4 to find an idempotent measure λ and $y \in G$ such that $\nu = \lambda * y$. Then $\frac{1}{2} < \nu(HVx) = \lambda(HVxy^{-1})$. We claim that $S_\lambda \subset HVVH$. Indeed, given an arbitrary point $z \in S_\lambda$ use the S_λ-invariance of λ to conclude that $\lambda(zHVxy^{-1}) = \lambda(HVxy^{-1}) > 1/2$, which implies that the intersection $zHVxy^{-1} \cap HVxy^{-1}$ is non-empty which yields $z \in HVxy^{-1}(HVxy^{-1})^{-1} = HVVH$. The inequality $\lambda(HVxy^{-1}) > 1/2$ implies that $HVxy^{-1}$ intersects S_λ and hence the set $HVVH$. Then $y \in HV^{-2}HHVx$ and $S_\nu = S_\lambda * y \subset HV^2HHV^{-2}HVx \subset Ux^{-1}x = U$, which implies that $\nu \in \mathrm{supp}^{-1}(U^+)$. This completes the proof of the continuity of the map $\mathrm{supp} : \mathrm{Reg}(P(G)) \to \mathrm{Reg}(\exp(G))$.

The proof of the continuity of the inverse map

$$\mathrm{supp}^{-1} : \mathrm{Reg}(\exp(G)) \to \mathrm{Reg}(P(G))$$

is even more involved. We first establish the continuity of this map under an additional assumption that the topological group G is first-countable. In this case G is metrizable and so are the spaces $\exp(G)$ and $P(G)$. So the continuity of supp^{-1} can be established by means of convergent sequences. Let $(K_n)_{n=1}^\infty \subset \mathrm{Reg}(\exp(G))$ be a sequence of compact subsets of G converging to a set $K_0 \in \mathrm{Reg}(\exp(G))$. For every $n \in \omega$ we find a compact subgroup H_n of G and a point $x_n \in G$ with $K_n = H_n x_n$. Denote the Haar measure on the compact group H_n by λ_n. It follows that $\mathrm{supp}^{-1}(K_n) = \lambda_n * x_n$ for all $n \in \omega$. Hence we must prove that the sequence of measures $(\lambda_n * x_n)_{n=1}^\infty$ converges to the measure $\lambda_0 * x_0$ in $P(G)$. Shifting this sequence by x_0^{-1} from the left, we may assume that x_0 is the neutral element of G.

Suppose to the contrary that the sequence $(\lambda_n * x_n)$ does not converge to $\lambda_0 = \lambda_0 * x_0$ and replacing (K_n) by a suitable subsequence, we may assume that λ_0 is not even a cluster point of the sequence $(\lambda_n * x_n)_{n=1}^\infty$. The convergence $H_n x_n \to H_0 \ni 1$ implies the existence of a sequence $(h_n)_{n=1}^\infty$ such that each $h_n \in H_n$ and $h_n x_n \to 1$. Since $Hx_n = H_n h_n x_n$, we can conclude that H_n tends to the group H_0.

The convergences $K_n \to K_0 = H_0$ and $H_n \to H_0$ imply that the union $K = \bigcup_{n\in\omega}(K_n \cup H_n)$ is compact and so is the space $P(K)$ containing the measures $\lambda_n * x_n$, $n \in \omega$. The compactness of $P(K)$ implies that the sequence $(\lambda_n * x_n)$ has a convergent subsequence. By replacing $(\lambda_n * x_n)$ by this subsequence, we may assume that the sequence $(\lambda_n * x_n)_{n=1}^\infty$ converges to some measure μ. We claim that $\mathrm{supp}(\mu) = K_0$. The inclusion $\mathrm{supp}(\mu) \subset K_0$ follows from the fact that for every closed neighborhood $O(K_0)$ of K_0 in G there is n_0 such that $K_n \subset O(K)$ and hence $\lambda_n * x_n \in P(O(K_0))$ for all $n \ge n_0$, which implies $\mu = \lim_{n\to\infty} \lambda_n * x_n \in P(O(K_0))$ and $\mathrm{supp}(\mu) \subset O(K_0)$. The inclusion $K_0 \subset \mathrm{supp}(\mu)$ follows from the convergences $\lambda_n * x_n \to \mu$ and $\mathrm{supp}(\lambda_n * x_n) = K_n \to K_0$. Therefore, $\mathrm{supp}(\mu) = K_0 = H_0$.

We claim that μ is the Haar measure on the group H_0. Since the Haar measure on H_0 is the unique right invariant measure, it suffices to check that for every bounded continuous real-valued function $f : G \to \mathbb{R}$ and every $a \in H_0$ we get $\mu(f) = \mu(f_a)$ where $f_a(x) = f(xa^{-1})$ for $x \in G$. Take any $\varepsilon > 0$. Observe that the function set $\mathcal{F} = \{f_b|K : b \in K\}$ is compact in the Banach space $C(K)$ of continuous functions on K. The convergence $\lambda_n * x_n \to \mu$ and the compactness of $\mathcal{F}$ imply the existence of a number $n_0 \in \mathbb{N}$ so large that $|\lambda_n * x_n(g) - \mu(g)| < \varepsilon/3$ for all $g \in \mathcal{F}$. The convergence $H_n \to H_0$ allows one to find a number $n \ge n_0$ and a point $b \in H_n$ so

close to the point $a \in H_0$ that in the Banach space $C(K)$ the difference $(f_a - f_b)|K$ has the norm $< \varepsilon/3$. Then $|\mu(f_a - f_b)| < \varepsilon/3$ and $\lambda_n(f_b) = \lambda_n(f)$ since the measure λ_n is H_n-invariant. Finally, we obtain

$$|\mu(f) - \mu(f_a)| \leq \\ |\mu(f) - \lambda_n(f)| + |\lambda_n(f) - \lambda_n(f_b)| + |\lambda_n(f_b) - \mu(f_b)| + |\mu(f_b) - \mu(f_a)| \leq \\ \frac{\varepsilon}{6} + 0 + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, μ, being a right-invariant measure on the group H_0 coincides with the Haar measure λ_0. This verifies the convergence $\lambda * x_n \to \lambda_0 * x_0$ and the continuity of the map $\mathrm{supp}^{-1} : \mathrm{Reg}(\exp(G)) \to \mathrm{Reg}(P(G))$ in case of first countable groups.

Now we consider the general case of an arbitrary topological group G. It is known that the sub-base of the topology of $P(G)$ consists of the sets $O(U, a) = \{\mu \in P(G) : \mu(U) > a\}$ where U runs over open subsets of G and a over real numbers. Thus to prove the continuity of the map $\mathrm{supp}^{-1} : \mathrm{Reg}(\exp(G)) \to \mathrm{Reg}(P(G))$ it suffices to check that for any such set $O(U, a)$ the preimage $\mathrm{supp}^{-1}(O(U, a) \cap \mathrm{Reg}(P(G)))$ is open in $\mathrm{Reg}(\exp(G))$. Take any element $K \in \mathrm{Reg}(\exp(G))$ with $\mu = \mathrm{supp}^{-1}(K) \in O(U, a)$. It follows that $\mu(U) > a$. Since the measure μ is Radon, there is a compact subset $C \subset U$ with $\mu(C) > a$. The compactness of C allows one to find a neighborhood $V = V^{-1}$ of the neutral element of the group G such that $CV \subset U$.

By [15, 2.3] there is a continuous homomorphism $h : G \to \tilde{G}$ onto a first countable group such that $h^{-1}(\tilde{V}) \subset V$ for some neighborhood $\tilde{V}$ of the neutral element of the group $\tilde{G}$. Let $Ph : P(G) \to P(\tilde{G})$ denote the map between the spaces of measures induced by the homomorphism h. Let $\tilde{K} = h(K)$, $\tilde{C} = h(C)$ and $\tilde{\mu} = Ph(\mu)$. It follows that $\tilde{\mu}(\tilde{C}\tilde{V}) \geq \tilde{\mu}(\tilde{C}) > a$ and thus $\tilde{\mu} \in O(\tilde{C}\tilde{V}, a)$. Now the continuity of the map $\mathrm{supp}^{-1} : \mathrm{Reg}(\exp(\tilde{G})) \to \mathrm{Reg}(P(\tilde{G}))$ yields an open neighborhood $\tilde{\mathcal{U}} \subset \mathrm{Reg}(\exp(\tilde{G}))$ of $\tilde{K}$ such that $\mathrm{supp}^{-1}(\tilde{\mathcal{U}}) \subset O(\tilde{C}\tilde{V}, a)$. The homomorphism $h : G \to \tilde{G}$ induces a continuous map $\exp(h) : \exp(G) \to \exp(\tilde{G})$ between the hyperspaces of G and $\tilde{G}$. Then the set $\mathcal{U} = \exp(h)^{-1}(\tilde{\mathcal{U}}) \cap \mathrm{Reg}(\exp(G))$ is an open neighborhood of K in $\mathrm{Reg}(\exp(G))$. We claim that $\mathrm{supp}^{-1}(\mathcal{U}) \subset O(U, a)$. Indeed, take any $K' \in \mathcal{U}$ and let $\mu' = \mathrm{supp}^{-1}(K')$ be the shifted Haar measure in K'. Let $\tilde{K}' = h(K')$ and $\tilde{\mu}' = Ph(\mu')$. It follows that $h(K') \in \tilde{\mathcal{U}}$ and thus $\tilde{\mu}' = \mathrm{supp}^{-1}(h(K')) \in O(\tilde{C}\tilde{V}, a)$, which means that $\tilde{\mu}'(\tilde{C}\tilde{V}) > a$. Now observe that $h^{-1}(\tilde{C}\tilde{V}) \subset Ch^{-1}(\tilde{V}) \subset CV \subset U$, which implies that $\mu'(U) \geq \mu'(h^{-1}(\tilde{C}\tilde{V})) = \tilde{\mu}'(\tilde{C}\tilde{V}) > a$. This means that $\mu' \in O(U, a)$. □

The following corollary establishes the first part of Proposition 1.3. The second part of that proposition follows from Theorem 2 of [5].

Corollary 2.6. *Let G be a topological group. Then a topological regular semigroup S can be embedded into the hypersemigroup $\exp(G)$ if and only if S can be embedded into the convolution semigroup $P(G)$.*

Proof. If $S \subset \exp(G)$ is a regular subsemigroup, then $S \subset \mathrm{Reg}(\exp(G))$ and $\mathrm{supp}^{-1}(S)$ is an isomorphic copy of S in $P(G)$ according to Propositions 2.5. Conversely, if $S \subset P(G)$ is a regular subsemigroup, then its image $\mathrm{supp}(S)$ is an isomorphic copy of S in $\exp(G)$. □

References

[1] T. Banakh, V. Gavrylkiv, *Algebra in superextensions of groups, II: cancelativity and centers*, Algebra Discrete Math. No. 4 (2008), to appear.

[2] T. Banakh, V. Gavrylkiv, *Algebra in the superextensions of groups, III: minimal left ideals*, preprint (arXiv:0805.1583).

[3] T. Banakh, V. Gavrylkiv, *Algebra in the superextensions of groups, IV: representation theory*, preprint (arXiv:0811.0796).

[4] T. Banakh, V. Gavrylkiv, O. Nykyforchyn, *Algebra in superextensions of groups, I: zeros and commutativity*, Algebra Discrete Math. No. 3 (2008), 1–29.

[5] T. Banakh, O. Hryniv, *Embedding topological semigroups into the hyperspaces over topological groups*, Acta Univ. Carolinae, Math. et Phys. **48**:2 (2007), 3–18.

[6] S.G. Bershadskii, *Imbeddability of semigroups in a global supersemigroup over a group*, Semigroup Varieties and Semigroups of Endomorphisms, Leningrad. Gos. Ped. Inst., Leningrad, 1979, pp. 47–49.

[7] R.G. Bilyeu, A. Lau, *Representations into the hyperspace of a compact group*, Semigroup Forum **13** (1977), 267–270.

[8] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups*, Vol. 1, Math. Surv., No.7, Amer. Math. Soc., Providence, R.I., 1964.

[9] V. Gavrylkiv, *Right-topological semigroup operations on inclusion hyperspaces*, Matem. Studii **29**:1 (2008), 18–34.

[10] H. Heyer, *Probability Measures on Locally Compact Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94, Springer-Verlag, Berlin-New York, 1977.

[11] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Amer. Math. Soc., Providence, R.I., 2005.

[12] M. Petrich, *Introduction to Semigroups*, Charles E. Merrill Publishing Co., Columbus, Ohio, 1973.

[13] J.S. Pym, *Idempotent measures on semigroups*, Pacific J. Math. **12** (1962), 685–698.

[14] A. Teleiko, M. Zarichnyi, *Categorial Topology of Compact Hausdorff Spaces*, VNTL Publ., Lviv, 1999.

[15] M. Tkačenko, *Introduction to topological groups*, Topology Appl. **86**:3 (1998), 179–231.

[16] A. Tortrat, *Lois de probabilité sur un espace topologique complètement régulier et produits infinis à termes indépendants dans un groupe topologique*, Ann. Inst. H. Poincaré Sect. B **1** (1964/1965), 217–237.

[17] V. Trnkova, *On a representation of commutative semigroups*, Semigroup Forum **10:3** (1975), 203–214.

[18] J.G. Wendel, *Haar measure and the semigroup of measures on a compact group*, Proc. Amer. Math. Soc. **5** (1954), 923–929.

Instytut Matematyki, Akademia Świętokrzyska, Kielce, Poland
and Department of Mathematics, Ivan Franko National University of Lviv, Lviv, Ukraine
E-mail address: tbanakh@yahoo.com

Institute of Mathematics, Physics and Mechanics, and Faculty of Education, University of Ljubljana, P.O.B. 2964, Ljubljana, 1001, Slovenia
E-mail address: matija.cencelj@guest.arnes.si

Department of Mathematics, Ivan Franko National University of Lviv, Lviv, Ukraine
E-mail address: olena_hryniv@ukr.net

Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P.O.B. 2964, Ljubljana, 1001, Slovenia
E-mail address: dusan.repovs@guest.arnes.si